1 Integrating Term by Term

**Theorem 1** Consider the power series

\[ f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad |z| < R (R \neq 0) \]

Let \( C \) be a simple piecewise smooth curve which lies inside the circle of convergence. Then we can **integrate the power series term by term**:

\[ \int_C \left( \sum_{n=0}^{\infty} a_n z^n \right) \, dz = \sum_{n=0}^{\infty} a_n \int_C z^n \, dz \quad (1) \]

**Proof.** The function \( f(z) \) defined by the power series is continuous on \( C \), so the integrals in (1) are well-defined. We need to show that

\[ \lim_{n \to \infty} \left| \int_C \left[ f(z) - \sum_{k=0}^{n} a_k z^k \right] \, dz \right| = 0 \quad (2) \]

Since \( C \) lies inside the circle of convergence, the series converges uniformly on \( C \) to \( f(z) \). For any \( \epsilon \), there is an \( N(\epsilon) \) so that, for all \( z \) on \( C \),

\[ n \geq N(\epsilon) \Rightarrow \left| f(z) - \sum_{k=0}^{n} a_k z^k \right| < \epsilon \]

By the triangle inequality for integrals and the above inequalities, for \( n \geq N \),

\[ \left| \int_C \left[ f(z) - \sum_{k=0}^{n} a_k z^k \right] \, dz \right| \leq \epsilon \cdot \text{(length of } C) \]

Since \( \epsilon \) is arbitrary, the limit in (2) is zero. \( \blacksquare \)